数理解析研究所講究録
第 862 巻 1994 年 146-170

INCOMPRESSIBLE FLUIDS ON THREE LEVELS: HYDRODYNAMIC, KINETIC, MICROSCOPIC*.

R. ESPOSITO[1] AND R. MARRA[2]

[1]Dipartimento di Matematica, Università di Roma Tor Vergata.
[2]Dipartimento di Fisica, Università di Roma Tor Vergata.

We present an overview of the incompressible Navier-Stokes flow as scaling limit starting from microscopic, kinetic and hydrodynamical descriptions. We also consider external forces and boundary conditions to include phenomena like the thermal convection in the scheme. The analysis is carried out both in the time-dependent and stationary case and we also discuss some rigorous results related to the scaling limit from the Boltzmann equation.

1. INTRODUCTION

The behavior of a fluid can be described on several scales: the macroscopic (hydrodynamical) scale, the mesoscopic (kinetic) scale and the microscopic scale. It is possible to connect them via suitable space-time scalings, so that each of the reduced descriptions is exact in suitable regimes. The simplest example of this is the Euler fluid which can be obtained as a scaling limit from microscopic (Newton) equations and from kinetic (Boltzmann) equations, by scaling both space and time by a factor ε^{-1} and taking the limit as $\varepsilon \to 0$. Of course the last one is confined to the case of perfect gas because of the low density assumption which is the basic assumption of the Boltzmann equation. The connection can be established on a rigorous mathematical basis at least with some technical assumptions (see for example [1], [2]).

The presence of viscosity and heat conduction effects makes above picture more complex. The main reason is in our opinion that there is no natural scale invariance in the equations for a viscous heat conducting fluid, like for the Euler fluid. Therefore, it is not possible

Work partially supported by MURST, INFN and CNR-GNFM.

to prove in general that such equations are exact in a suitable scaling limit. We can only consider a few examples in which special symmetries or specific regimes restore a kind of scale invariance. This paper is devoted to a short review of the examples for which rigorous results are available (to our knowledge).

The incompressible flow of a viscous fluid is probably the simplest example of the possibility of restoring the scale invariance. It is well known (see for example [3]) that the incompressible regime for the Euler and Navier-Stokes equations can be obtained in the limit in which the Mach number (Ma) goes to zero. It has been observed (see [4]) that if the Knudsen number Kn (proportional to the mean free path) and the Mach number go to zero with the same speed in ε, the Reynolds number Re (which is the ratio between the transport and viscosity effects) stays finite. This can be interpreted in terms of scaling as follows: by scaling space and time as ε^{-1} and ε^{-2} and the velocity field as ε, the incompressible Navier-Stokes equations are unchanged. Starting from this, in [5] (see also [6]) it has been proved that the incompressible Navier-Stokes equations are limit of the Boltzmann equation under above scaling, under suitable technical assumptions. Extensions of this results have been provided in [7], [8]. Precise statements will be given in Section 3.

The same kind of results could be in principle obtained starting from the Newton equations for a Hamiltonian system of particles (with ε now proportional to the ratio between the range of interaction and the macroscopic length). In this case however, we have only a formal proof of the result (see [9]) because of the enormous difficulties to be overcome to get rigorous proofs of convergence from deterministic microscopic systems. A rigorous result is instead available for a related problem, i.e. the derivation of the viscous Burgers equation from the simple exclusion model (see [10]). The formal derivation of the incompressible Navier-Stokes equation will be shortly discussed in Section 4.

As said before, the same question arises naturally in the framework of the compressible Navier-Stokes equations, where the incompressible limit can be reinterpreted as a scaling limit and results of convergence have been obtained, among the others, by [11]. This will be discussed in Section 2.

Interesting phenomena are related to the presence of an external force. For this reason we will also include in our setup an external force F acting on the fluid. A simple dimensional analysis shows that the force F has to be scaled as ε^3 to be consistent with the incompressible regime, i.e. to get a finite force term in the equation for the velocity field. However, the case of a conservative force is special from this point of view. In fact, if the mass density tends to a constant, larger conservative forces are compatible with the incompressible regime because they can be compensated by suitable contributions to the pressure. Due to the fluctuations of the mass density, buoyancy effects arise, which are controlled by a new parameter, the Rayleigh number Ra. Hence the force has to be scaled so to keep Ra finite. It turns out that the conservative force is to be scaled as ε^2. Since the most important application of this remark is related to the Benard problem, where the fluid is confined between two plates at slightly different temperatures, we will also include the boundary conditions in our discussion. We consider the walls at fixed temperature and no-slip boundary conditions for the velocity. At kinetic and microscopic level we will model them assuming that each particle colliding with the boundary is reflected with a velocity at random, distributed according to the equilibrium distribution at the temperature of the boundary. This is probably the simplest example of thermal wall, but of course more general conditions could be allowed, at the prize of more technicalities, so we will not discuss such general boundary conditions. Actually, in the kinetic case, for which we can give a rigorous proof, we have to modify above prescription on the boundary for technical reasons, introducing the *bulk* boundary conditions discussed at the end of Section 3.

2. Hydrodynamic description.

We consider a heat-conducting viscous fluid in a domain $\Omega \subset {}^d$. We fix $d = 3$ from now on, for sake of definiteness. Ω will be in the following either ^{3} or the 3-dimensional torus ^{3} of side 1 or the slab ${}^2 \times [-1,1]$. In the last case we have a boundary $\partial\Omega \neq \emptyset$, on which we specify the temperature, the pressure and the velocity field as:

$$T(1) = T_+, \qquad T(-1) = T_- = T_+ + \delta T \geq T_+,$$
$$P(1) = P_+, \qquad U(-1) = 0 = U(1).$$

We introduce a space scale parameter ε so that Ω is the image after rescaling of a domain

$\Omega^{(\varepsilon)}$ obtained from Ω with an expansion of a factor ε^{-1}. Hence in the case of the slab the meaning of ε is the inverse of the size of the slab in the original units, while in the case without boundary it is basically the scale on which the initial datum changes. We also consider the limit of very small velocities, i.e. we assume $U = \varepsilon u$. Moreover we consider a force $\varepsilon^2 G$ with $G = -\nabla V$. In this conditions, to see finite displacements from the initial data one has to wait for times of ordet $\varepsilon^{-2}t$. Therefore, the scaling considered is

$$x \to \varepsilon^{-1}x, \quad t \to \varepsilon^{-2}t, \quad u \to \varepsilon u, \quad G \to \varepsilon^2 G \tag{2.1}$$

We write the compressible Navier-Stokes equations in rescaled variables as

$$\partial_t \rho + \mathrm{div}(\rho u) = 0, \tag{2.2}$$

$$\rho(\partial_t u + u \cdot \nabla u) = \eta \Delta u - \varepsilon^{-2}\nabla P + \varepsilon^{-1}\rho G + \zeta \nabla \mathrm{div} u, \tag{2.3}$$

$$\rho T(\partial_t s + u \cdot \nabla s) = \kappa \Delta T + \varepsilon^2 \eta (\nabla u)^2. \tag{2.4}$$

Here the viscosity coefficients η and ζ and the conductivity κ, which depend on the temperature, for shortness are considered constant in this approximation, in view of the limit we are going to consider, because the result would be the same. s and P are the specific entropy and the pressure, which are given by some state equations. We consider for sake of simplicity the case of a perfect gas $P = c\rho T$ and $s = c_v \log T - c \log \rho$ with $c > 0$ the constant of the ideal gases and $c_v = (3/2)c$ the specific heat at constant volume. This choice allows also to compare the results of this section with the ones of section 3, where the state equation coming out from the Boltzmann equation is exactly the one for ideal gas. The only external force we are really interested in is the gravity, so we assume $V = g(z+1)$, with $z \in [-1,1]$ the third component of $\underline{x} = (x,y,z)$ and $g > 0$ the acceleration of gravity. In this case a natural parameter is the Rayleigh number Ra defined as $Ra = [(\kappa\nu)^{-1}(gL^3\delta T/T_+)]^{1/2}$. The scaling considered actually leaves this number invariant and this justifies the choice of our scaling of the force. In fact a way to obtain the rescaled NSE is to introduce dimensionless variables

$$t' = t\nu L^{-2}, \quad \underline{x}' = \underline{x}L^{-1},$$

$$T' = (T - T_+)(\delta T)^{-1}, \quad u' = (gL\delta T/T_+)^{-\frac{1}{2}} u.$$

and rewrite the NSE in dimensionless form.

In order to be consistent with the rescaling we have to consider initial and boundary data suitably dependent on ε. In fact, we assume at time zero:

$$T(x,0) = T_+ + \varepsilon\hat{\theta}(x,0), \qquad \rho(x,0) = \rho_+ + \varepsilon\ \hat{r}(x,0)$$

with ρ_+ the given constant density. On the boundary we assume that $(T_+)^{-1}\delta T = 2\lambda\varepsilon$. We expect that at positive times we still have

$$T(x,t) = T_s(x) + \varepsilon\theta(x,t), \qquad \rho(x,t) = \rho_s(x) + \varepsilon r(x,t) \tag{2.5}$$

where, if $\partial\Omega = \emptyset$, then $T_s = T_+$ and $\rho_s = \rho_+$ are some positive constants, while, in the case of the slab, ρ_s and T_s are the solution of the stationary problem

$$\frac{d}{dz}P_s = -\varepsilon g\rho_s, \qquad \Delta T_s = 0, \tag{2.6}$$

with $P_s = c\rho_s T_s$ and boundary conditions

$$T_s(-1) = T_-, \quad T_s(1) = T_+, \quad \rho(1) = P_+/(cT_+) = \rho_+.$$

In other words r and θ represent the fluctuations with respect the density and temperature profiles corresponding to the hydrostatic equilibrium given by

$$T_s = T_+(1 + \varepsilon\lambda(1-z)), \ \ \rho_s = \rho_+(1 + \varepsilon\lambda(1-z))^{\frac{g}{c\lambda T_+}-1} \tag{2.7}$$

In particular this means that $\theta(-1) = 0 = \theta(1)$. Substituting (2.5) in (2.2) we get

$$-\rho_+ \mathrm{div} u = \varepsilon[\partial_t r + \mathrm{div}(ru)] + \mathrm{div}[u(\rho_s - \rho_+)]$$

We assume that u, r, θ converge, for $\varepsilon \to 0$, to finite limits (that we still call u, r, θ) in a strong enough sense. Since $(\rho_s - \rho_+)$ is of order ε, we get in the limit

$$\mathrm{div} u = 0 \tag{2.8}$$

Taking into account (2.5) and (2.8), equation (2.3) gives,

$$-\nabla[P_s + \varepsilon\tilde{P}_1] + \varepsilon[\rho_s + \varepsilon r]G = O(\varepsilon^2)$$

where we used $P = P_s + \varepsilon\tilde{P}_1 + \varepsilon^2\tilde{P}_2 + O(\varepsilon^3)$ with $\tilde{P}_1 = \rho_+\theta + T_+r$. Then, using (2.6), in the limit we get

$$\nabla\tilde{P}_1 = 0, \tag{2.9}$$

that is nothing but the Boussinesq condition

$$\rho_+\nabla\theta + T_+\nabla r = 0 \tag{2.10}$$

In consequence

$$\rho_+\theta + T_+r = a \tag{2.11}$$

with a independent on $\underline{x}$ and t. The constant does not depend on t because the total pressure is constant in time at the boundary. Indeed $a = \tilde{P}_1(1) = 0$, since $P(1) = P_s(1) = P_+$. In the case $\partial\Omega = \emptyset$ this follows from a slightly different argument. Condition (2.11) is assumed as a "state equation" in the usual discussions of the Boussinesq approximation (see [12], [13]), while in this approach it is just a consequence of the scaling limit.

Taking the limit of the momentum equation (2.3) one gets:

$$\rho_+(\partial_t u + u\cdot\nabla u) = \eta\Delta u - \nabla\tilde{P}_2 + rG.$$

Using the Boussinesq condition (2.11) it becomes

$$\rho_+(\partial_t u + u\cdot\nabla u) = \eta\Delta u - \nabla p - \alpha\rho_+\theta G \tag{2.12}$$

where $\alpha = T_+^{-1}$ is the coefficient of thermal expansion and $p = \tilde{P}_2$ is the unknown pressure of the incompressible problem.

Finally the equation for the entropy (2.4) gives

$$\rho c_v D_t T - cTD_t\rho = \kappa\Delta T + O(\varepsilon^2)$$

with $D_t \equiv \partial_t + u\cdot\nabla$. The state equation implies that $cTD_t\rho = -\rho cD_tT + D_tP$. By (2.11) $D_tP = -\varepsilon\rho_+u_zg + O(\varepsilon^2)$. Hence we get the following equation for θ

$$c_p(\partial_t\theta + u\cdot\nabla\theta) + u_z(g - \lambda c_pT_+) = \rho_+^{-1}\kappa\Delta\theta \tag{2.13}$$

with $c_p = c_v + c = (5/2)c$. The boundary conditions associated to the system (2.8), (2.12), (2.13) are:

$$u(-1) = 0 = u(1), \qquad \theta(-1) = 0 = \theta(1)$$

In the case $\partial\Omega = \emptyset$, $G = 0$ and $\lambda = 0$, the system (2.8), (2.12) is just the usual incompressible Navier-Stokes equation in ^{3} or ^{3} and (2.13) is the heat equation.

If $\partial\Omega \neq \emptyset$ it is more convenient to rewrite (2.12) (2.13) in terms of the variable $\tilde{\theta} = \theta - ({c_p}^{-1}g - \lambda T_+)(1-z)$ as

$$c_p D_t \tilde{\theta} = \rho_+^{-1} \kappa \Delta \tilde{\theta} \tag{2.14}$$

with boundary conditions

$$\tilde{\theta}(1) = 0, \quad , \tilde{\theta}(-1) = -2(c_p^{-1}g - \lambda T_+). \tag{2.15}$$

Equation (2.12) becomes

$$\rho_+(\partial_t u + u \cdot \nabla u) = \eta \Delta u - \nabla \tilde{p} - \alpha \rho_+ \tilde{\theta} G \tag{2.16}$$

with

$$\tilde{p} = p + \alpha \rho_+ g({c_p}^{-1}g - \lambda T_+)(1-z)^2/2, \tag{2.17}$$

The system (2.8), (2.16), (2.14) with the boundary conditions (2.15) and no-slip conditions on u, differs from the usual Oberbeck-Boussinesq (OBE) equations (see [12], [14]) because of the term proportional to g in (2.15) and of the quadratic term in g in (2.17). In the usual experimental conditions (see [15]) such terms are much smaller than the others, thus one can neglect the effect of the variation of the density with the altitude due to the gravitational force, namely consider $P_s \sim P_+$. This effect becomes relevant in situations when the force g is much bigger then the earth's gravity. In this way we recover the usual OBE.

Above discussion is only formal because it assumes the convergence of u, r, θ to a limit when $\varepsilon \to 0$. This is far from being obvious and has been actually proved only in special cases. In [11] it is considered an isentropic flow, with $P = c\rho^\gamma$, $c > 0$, $\gamma \geq 1$. Assuming $\Omega = {}^d$, $G = 0$ and initial data in some Sobolev space, the convergence is proved in a suitable H_s-norm and it is sufficient to make previous arguments rigorous. For the case of the slab with $G \neq 0$ there is no proof available, but we believe that above results can be extended with minor changes. In fact the presence of the boundary should not produce additional problems due to boundary layer corrections because the scaling is such that the coefficients of the dissipative terms stays finite in this limit. A detailed analysis of these problems is beyond the purposes of this presentation.

3. Kinetic description

In this section we examine the problem of the previous section from the kinetic point of view. Most of the discussion is given in the case of the slab with an external potential force. The case $\partial\Omega = \emptyset$ and $G = 0$ easily follows. We consider the Boltzmann equation (BE) for a gas between parallel planes. We keep the notations of Sect.2. To model the hydrodynamic boundary conditions we choose the so called Maxwellian boundary conditions: when a particle hits the edges of the channel ($z = -1$ or $z = 1$) it is diffusely reflected with a velocity distributed according to a Maxwellian with zero mean velocity and prescribed temperatures T_- and T_+ respectively. In the language of classical kinetic theory this means that the *accommodation coefficient* equals one. Above prescription implies the impermeability of the walls, namely no mass flux across the boundary is allowed. We introduce as scale parameter ε the Knudsen number. The size of the channel is $2\varepsilon^{-1}$, hence in rescaled variables $z \in [-1, 1]$. Let us take for simplicity periodic conditions in direction x, y, and call $\Omega = \{\underline{x}, v : (x,y) \in {}^2, z \in [-1,1], v \in {}^3\}$. The BE rescaled according to the scaling (2.1) is

$$\partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla f^\varepsilon + G \cdot \nabla_v f^\varepsilon = \varepsilon^{-2} Q(f^\varepsilon, f^\varepsilon) \tag{3.1}$$

We confine ourselves to the collision operator Q for hard spheres [16]. The initial condition is

$$f^\varepsilon(0, \underline{x}, v) = f_0(\underline{x}, v) \quad z \neq \pm 1, \tag{3.2}$$

but we shall see that it cannot be given arbitrarily without a detailed analysis of the initial layer which we skip in this paper. The precise form of f_0 will be made clear below. The boundary conditions are:

$$f^\varepsilon(t,x,y,-1,v) = \alpha_- \overline{M}_-(v), \quad v_z > 0, \ t > 0 \tag{3.3}$$

$$f^e(t,x,y,1,v) = \alpha_+ \overline{M}_+(v), \quad v_z < 0, \ t > 0 \tag{3.4}$$

with

$$\overline{M}_\pm(v) = \frac{1}{2\pi T_\pm^2} e^{-v^2/2T_\pm}, \tag{3.5}$$

normalized so that $\int_{v_y 0} |v_y| \overline{M}_\pm(v)dv = 1$. Here and below we put the perfect gas constant, denoted by c in previous sections, equal to 1 and in consequence the constants c_v and c_p become the numbers 3/2 and 5/2 respectively.

The quantities $\alpha_\pm$ depend on the solution in such a way that the impermeability condition of the walls is assured:

$$\langle v_z f \rangle \equiv \int_3 v_z f dv = 0 \qquad \text{for } z = \pm 1, \tag{3.6}$$

where we have introduced the notation $\langle f \rangle = \int_3 f(v)dv$. Condition (3.6) and the normalisation of $\overline{M}_\pm$ imply:

$$\alpha_\pm = \pm \int_{v_z 0} v_z f(t,x,y,\pm 1, v)dv \tag{3.7}$$

Namely, $\alpha_\pm$ represent the outgoing (from the fluid) fluxes of mass in the direction z. From (3.6) the total mass stay constant in time and we put $\int_\Omega f^\varepsilon = m$. The macroscopic behavior is recovered in the limit $Kn \equiv \varepsilon$ going to zero. We expect that for ε small the behavior of the solution to the initial-boundary value problem is very close to the hydrodynamical one, in the sense that it can be described by a local Maxwellian with parameters which differ from constants by terms of order ε, so that these terms are solution of the OBE. At higher order there should be kinetic corrections and boundary layer corrections. Therefore we look for a solution of the form

$$f^\varepsilon = M + \varepsilon f_1 + \sum_{n=2}^{5} \varepsilon^n f_n + \varepsilon^3 R \tag{3.8}$$

where M is the global Maxwellian

$$M(\rho_+, 0, T_+; v) = \frac{\rho_+}{(2\pi T_+)^{3/2}} e^{-v^2/2T_+}.$$

If we put (3.8) in the BE (3.1) we see immediately that it has to be

$$2Q(M, f_1) \equiv \mathcal{L}f_1 = 0 \tag{3.9}$$

where $\mathcal{L}$ is the linearized Boltzmann operator. Since f_1 has to be in Null$\mathcal{L}$, which is 5-dimensional, (3.9) implies that it is a combination of the collision invariants $M\chi_i$ with $\chi_i(v) = 1, v_i, (v^2 - 3T_+)/2$, for $i = 0$, $i = 1,2,3$ and $i = 4$ respectively, suitably normalized to form an orthonormal set, in $L_2(M(v)^{-1}dv)$. Hence we have

$$f_1 = M\sum_{i=0}^{4} \chi_i t_i(t, \underline{x}) \equiv M\Big(\frac{\hat{r}}{\rho_+} + \frac{u\cdot v}{T_+} + \hat{\theta}\frac{v^2 - 3T_+}{2T_+^2}\Big). \tag{3.10}$$

The functions $t_i(t, \underline{x})$ and/or $\hat{r}, u, \hat{\theta}$ will satisfy equations to be determined. To write the conditions for f_n we decompose them in a part B_n, representing the bulk corrections, and boundary layer corrections $b_n^{\pm}$, sensibly different from 0 only near the boundary. B_n have to satisfy for $n = 2, \cdots, 5$

$$\partial_t B_{n-2} + v\cdot\nabla B_{n-1} + G\cdot\nabla_v B_{n-2} = 2Q(M, B_n) + \sum_{i+j=n} Q(B_i, B_j) \tag{3.11}$$

where $B_0 \equiv M$ and $B_1 \equiv f_1$. The boundary layer terms are obtained scaling back to microscopic coordinates around $z = \pm 1$. Denoting $z^{\pm} = \varepsilon^{-1}(1 \mp z)$ so that $z^{\pm} \in [0, 2\varepsilon^{-1}]$, the boundary layer corrections relative to the wall $z = \pm 1$, $b_n^{\pm}$, have to satisfy, for $n = 2\ldots 5$,

$$\begin{aligned}\partial_t b_{n-2}^{\pm} + v_z\frac{\partial b_n^{\pm}}{\partial z^{\pm}} + \hat{v}\cdot\hat{\nabla} b_{n+1}^{\pm} + G\cdot\nabla_v b_{n-2}^{\pm} = \\ \mathcal{L}b_n^{\pm} + \sum_{\substack{i,j\geq 1\\ i+j=n}} \Big[2Q(B_i, b_j^{\pm}) + Q(b_i^{\pm}, b_j^{\pm}) + Q(b_i^{\mp}, b_j^{\mp})\Big],\end{aligned} \tag{3.12}$$

where we put $b_0^{\pm} = b_1^{\pm} = 0$ and $\hat{v} = (v_x, v_y)$, $\hat{\nabla} = (\partial_x, \partial_y)$. Finally the weakly nonlinear equation for the remainder is

$$\partial_t R + v\cdot\nabla R + G\cdot\nabla_v R = \frac{1}{\varepsilon^2}\mathcal{L}R + \frac{1}{\varepsilon}\mathcal{L}^{(1)}R + \mathcal{L}^{(2)}R + \varepsilon Q(R,R) + \varepsilon A \tag{3.13}$$

with

$$\mathcal{L}^{(1)}R = 2Q(f_1, R), \quad \mathcal{L}^{(2)}R = 2Q(\sum_{n=2}^{5}\varepsilon^{n-2}f_n, R) \tag{3.14}$$

and A given by

$$A = -\partial_t(f_4 + \varepsilon f_5) - v\cdot\nabla B_5 - \hat{v}\cdot\hat{\nabla} b_5^+ - \hat{v}\cdot\hat{\nabla} b_5^- -$$
$$G\cdot\nabla_v(f_4 + \varepsilon f_5) + \sum_{\substack{k,m\geq 1\\ k+m\geq 6}} \varepsilon^{k+m-6} Q(f_k, f_m). \tag{3.15}$$

The boundary conditions for these equations have to be chosen in such a way to satisfy (3.3)–(3.6) for f^ε. We are interested in the case $T_- > T_+$. Since $T_- = T_+(1+2\varepsilon\lambda)$ it is easy to satisfy (3.3) and (3.4) up to the first order in ε, because M is already a Maxwellian and temperature and velocity field are chosen to fit with $\overline{M}_\pm$. The density only has to be adjusted. From the second order on we have to use boundary layer terms to fit boundary conditions. In fact, as we will see later, $B_n, n\geq 2$ do not reduce to $\alpha_n^\pm \overline{M}_\pm$. The idea is to introduce at one of the boundaries, say $z=1$, the correction b_2^+ to so that $B_2 + b_2^+$ is proportional to $\overline{M}_+$ for $v_z < 0$. On the other hand, the same has to be done in $z=-1$ and f_2 is modified by b_2^- also. This changes again f_2 in $z=1$ by non Maxwellian terms. However, since b_2^- decays exponentially fast, the modification is exponentially small in ε^{-1}. Therefore we impose to the f_n the following boundary conditions:

$$f_n(t,x,y,-1,v) = \alpha_n^- \overline{M}_-(v) + \gamma_{n,\varepsilon}^-(v), \qquad v_z > 0,\ \ t>0$$
$$f_n(t,x,y,1,v) = \alpha_n^+ \overline{M}_+(v) + \gamma_{n,\varepsilon}^+(v), \qquad v_z < 0,\ \ t>0 \tag{3.16}$$

with the functions $\gamma_{n,\varepsilon}^\pm(v)$ exponentially small in ε^{-1} and such that $\langle\gamma_{n,\varepsilon}^\pm v_z\rangle = 0$, to be specified later. Moreover

$$\alpha_n^\pm = \pm\int_{v_z 0} v_z f_n(t,x,y,\pm 1, v)dv \tag{3.17}$$

Finally, to fulfil (3.3) and (3.4) we impose the following conditions on R:

$$R(t,x,y,-1,v) = \alpha_R^- \overline{M}_-(v) - \sum_{n=2}^{5}\varepsilon^{n-3}\gamma_{n,\varepsilon}^- \qquad v_z > 0,\ \ t > 0 \tag{3.18}$$

$$R(t,x,y,1,v) = \alpha_R^+ \overline{M}_+(v) - \sum_{n=2}^{5}\varepsilon^{n-3}\gamma_{n,\varepsilon}^+ \qquad v_z < 0,\ \ t > 0 \tag{3.19}$$

The initial conditions for $R(0,\underline{x},v)$ are chosen to be $R(0,\underline{x},v) = 0,\ \ z\neq\pm 1$ for simplicity. The initial values for the f_n's are partly determined by the procedure below, so that only their hydrodynamical part can be assigned. To remove such restrictions one has to include an initial layer analysis, which we skip to make the presentation shorter.

The equations for the f_n are coupled in a complicated way and have to be solved in the proper sequence, which we now outline. The hydrodynamical part of the bulk terms is determined by the solvability conditions for (3.11), that we get multiplying (3.11) by $\chi_i, i = 0 \dots 4$, integrating on velocity and using the fact that $\langle Q(f,g)\chi_i\rangle = 0$. The solvability condition for (3.11) with $n = 2$ is

$$\langle \chi_i [v \cdot \nabla f_1 + G \cdot \nabla_v M]\rangle = 0, \tag{3.20}$$

because the Maxwellian M does not depend on x and t. This is equivalent to

$$\mathrm{div} u = 0, \qquad \rho_+ \nabla \hat{\theta} + T_+ \nabla \hat{r} = \rho_+ G. \tag{3.21}$$

The first one is the usual vanishing divergence condition. The second one becomes the Boussinesq condition (2.10), when one defines $\theta = \hat{\theta} - \lambda(1-z)T_+$ and $r = \hat{r} - \rho_+(g/T_+ - \lambda)(1-z)$. Once (3.21) are satisfied, we can deduce from (3.11) with $n = 2$ the following expression for B_2:

$$B_2 = \mathcal{L}^{-1}\Big[v \cdot \nabla f_1 + G \cdot \nabla_v M - Q(f_1, f_1)\Big] + M \sum_{i=0}^{4} \chi_i \, t_i^{(2)}(t, \underline{x}) \tag{3.22}$$

The solvability condition for (3.11) with $n = 3$ is

$$\langle \chi_i [\partial_t f_1 + G \cdot \nabla_v f_1 + v \cdot \nabla B_2]\rangle = 0, \tag{3.23}$$

and this produces the equations for u and θ. Let us fix $i = 1, 2, 3$ in (3.23). Then the first term gives the time derivative of $\rho_+ u$. The last one reduces to $-G\hat{r}$ integrating by parts. Finally we write

$$\langle v \otimes v B_2\rangle = \langle [v \otimes v - \frac{v^2}{3}] B_2\rangle + \langle \frac{v^2}{3} B_2\rangle$$

The first term has been computed for example in [17] and gives rise to the dissipative term and the transport term in the equation (2.12), while the second one is the second order correction to the pressure P_2. The result is

$$\rho_+(\partial_t u + u \cdot \nabla u) = \nu \Delta u - \nabla P_2 + Gr.$$

Using the Boussinesq condition, the definitions of r and θ and the relation between P_2 and $\tilde{P}_2$ of section 2, we find (2.12) as in section 2, with ν given by

$$\nu = \langle (v \otimes v - \frac{v^2}{3}) \mathcal{L}^{-1} [M (v \otimes v - \frac{v^2}{3})] \rangle .$$

To get the equation for the temperature, it is convenient to replace in (3.23) χ_4 by $\hat{\chi}_4 = \frac{1}{2}[v^2 - 5T_+)]$. An easy computation using the Boussinesq condition provides:

$$\langle \frac{1}{2}[v^2 - 5T_+)] f_1 \rangle = \frac{5}{2}\rho_+ \left[\theta - z(\frac{2}{5}g - \lambda T_+)\right] + \text{const.},$$

$$\langle \frac{1}{2}[v^2 - 5T_+) G \cdot \nabla_v f_1 \rangle = \rho_+ u_z g,$$

$$\langle v \frac{1}{2}[v^2 - 5T_+)] B_2 \rangle = -\kappa \nabla\theta + \frac{5}{2}\rho_+ u(\theta + \lambda T_+(1 - z)).$$

Collecting above results together we get (2.13) with κ given by

$$\kappa = \langle v \frac{1}{2}(v^2 - 5T_+) \mathcal{L}^{-1}[M v \frac{1}{2}(v^2 - 5T_+)] \rangle . \tag{3.24}$$

Finally, eq (3.23) with $i = 0$ gives

$$\partial_t r = \operatorname{div} \underline{t}^{(2)}, \qquad \underline{t}^{(2)} = (t_1^{(2)}, t_2^{(2)}, t_3^{(2)}). \tag{3.25}$$

Summarizing, so far we have shown that, given a solution u, p, θ of OBE (2.8), (2.12) and (2.13), the coefficients t_i entering in the definition of f_1 are determined. Therefore, once initial and boundary conditions for OBE are specified, f_1 is completely determined as function of $(t, \underline{x}, v)$.

On the other hand the hydrodynamic part of B_2 is not yet determined, but, by (3.25), div $\underline{t}^{(2)}$ is determined in terms of r. Moreover, a combination of $t_0^{(2)}$ and $t_4^{(2)}$ contributes to the pressure p which is determined by the OBE, so that these parameters are not independent.

The non-hydrodynamic part of B_2 depends on the derivatives of r, u, θ which are in general different from zero on the boundaries. Therefore B_2 violates the boundary conditions and we need to introduce $b_2^{\pm}$ to adjust the boundary conditions. We choose b_2^- solving for any $t > 0$ the Milne problem

$$v_z \frac{\partial}{\partial_z} h = \mathcal{L} h, \quad \langle v_z h \rangle = 0$$

with boundary condition (in $z^- = 0$) given prescribing the incoming flux as the opposite of the non hydrodynamic part of B_2 in $z = -1$. Well known results [18], [19] tell us that the solution approaches, as $z^- \to \infty$ a function q_2^- in Null$\mathcal{L}$. Thus $b_2^- = h - q_2^-$ will go to zero at infinity exponentially in z^- and will be the boundary layer correction we are looking for.

In conclusion, in $z = -1$

$$f_2(t,x,y,,-1,v) = t^{(2)}(t,x,y,-1) + b_2^+(t,x,y,2\varepsilon^{-1}) - q_2^-, \quad v_z > 0, \quad t > 0$$

Since $t^{(2)}$ can be chosen arbitrarily on the boundaries we use it to compensate q_2^-. To satisfy the impermeability conditions we have to choose on the boundaries $t_3^{(2)} = 0$. The coefficients of the hydrodynamic part of B_2 for $i \neq 0$ will be determined by the compatibility condition for $n = 3$ that are the time-dependent non-homogeneous Stokes equations (linear second order differential equations) on a slab, together with the b. c. $t_i^{(2)} = q_{2i}^-$, $i = 1,2,4$. Then $t_0^{(2)}$ is found up to a costant that is chosen so that the total mass associated to f_2 vanishes. Finally we get

$$f_2(t,x,y,\pm 1, v_z \quad 0 \quad) = \alpha_2^\pm M_\pm + \gamma_{2,\varepsilon}^\pm, \quad \alpha_2^\pm = t_0^{(2)}(x,y,\pm 1) - q_2^{(\pm)}(0)$$

Iterating this procedure it is possible to find all f_n.

¿From the rigorous point of view, there are several results for the case $\partial\Omega = \emptyset$ and $G = 0$. In fact, in [5] it has been proved that for $\Omega = {}^d$, if $t_0 > 0$ is such that there is an unique solution to the incompressible Navier-Stokes equation (INS) with finite H_s norm for s large enough, then there is a solution to the Boltzmann equation of the form (3.8) with R bounded in the L_∞ norm. For $\Omega = {}^d$ and small enough initial data in [8] it is proven the existence of the solution of the Boltzmann equation and its convergence to the solution of the INS. In [7] it is considered the problem of the convergence of the Di Perna Lions weak solutions of the Boltzmann equation to the Lerey-Hopf weak solutions of the INS. The results is achieved only partially. In fact it is proven the convergence of the solution to some limit point, but the information available are not sufficient to show that the limit point solves the INS and some extra compactness has to be assumed, or obtained by suitable regularizations like discretization of the time variable.

In presence of the boundary, but still with $G = 0$, at least in the case of the slab, a result similar to the one in [5] can be obtained along the lines indicated above, combined with the method of dealing with the boundary layer correction presented in the papers [20], [21]. More general domains require a more accurate analysis of the boundary layer corrections.

The case $G \neq 0$ in the slab presents a new difficulty. In fact, the Milne problem for $b_4^{\pm}, b_5^{\pm}$, eq.s (3.12) for $n = 4, 5$, involves the derivatives with respect to v of the boundary layer term of $b_2^{\pm}$, which appear as a source term. Therefore when solving (3.12), $n = 2$, we need good properties of the v-derivative to apply the theorems and conclude about the exponential decay of $b_2^{\pm}$. Moreover v-derivatives appear also in the term A in the equation for the remainder, so that we need to control v-derivatives of $b_n^{\pm}$ for all n. At the moment we don't know how to get this result. From the physical point of view, what happens is the following [22]: in the free case (no collisions) the solution is discontinuous in velocity on the boundary. This discontinuity propagates in the gas and molecular collisions can attenuate it. Then the problem is how this discontinuity can propagate when collisions are present. Heuristic arguments show that if the boundary is convex, the discontinuity, propagating along the characteristics, enters the gas. On the other hand, if the body is concave the characteristic is tangent to the boundary and the effect does not appear in the gas. The case we are considering (parallel planes) is borderline and what is expected is that the discontinuity travel along the boundary without entering the fluid.

Note that, if the force is parallel to the plates, such a difficulty does not appear because there is no discontinuity in the velocities v_x and v_y. Actually a constant force parallel to the plates cannot be derived from a potential and hence it is not possible to compensate it by a pressure term. Therefore modifications of the hydrodynamical fields appear already at the zero order in ε, corresponding to compressible contributions. The stationary solutions in this case have been considered in [20] and [21], to which we refer for details. The incompressible flow is obtained instead by scaling the force as ε^3.

To avoid the previous difficulty we modify the boundary conditions, introducing what we call *bulk* boundary conditions (see below). This allows to separate the difficulties of the boundary layer from the difficulty in the bulk and show that the expansion is correct at least in the bulk as $\varepsilon \to 0$. The result is achieved by proving L_∞ estimates for the remainder. In this way we will get a rigorous proof of the hydrodynamic Boussinesq behavior in the bulk.

We consider the following modified boundary value problem: find a solution to the Boltzmann equation (3.1), with (3.3) and (3.4) replaced by the *bulk* boundary conditions

$$
\begin{aligned}
&f(t,x,y,-1,v) = \alpha_- \overline{M}_-(v) - \sum_{n=2}^{5} \varepsilon^n f_n(t,x,y,-1,v) \qquad v_z > 0,\ \ t > 0 \\
&f(t,x,y,+1,v) = \alpha_+ \overline{M}_+(v) - \sum_{n=2}^{5} \varepsilon^n f_n(t,x,y,+1,v) \qquad v_z < 0,\ \ t > 0,
\end{aligned} \tag{3.26}
$$

with

$$
\langle f_n(t,x,y,\pm 1,v)\chi_i(v)\rangle = 0 \qquad \text{for } n = 2,\dots,5 \text{ and } i = 1,\dots,4.
$$

$\alpha_\pm$ have still the meaning of outgoing fluxes of mass, if we require $\langle v_z f_n\rangle = 0$. In this way the solution is required to match a Maxwellian only up to the first order in ε, while the higher order terms are such that their non hydrodynamical part is fixed on the boundary by the bulk expansion, and they do not contribute to the velocity field and temperature at the boundary. This situation can be seen as the one produced by imaginary walls in the fluid, on which one fixes in some way the hydrodynamical fields, while for the rest they behave as part of the bulk. This is the reason why we call it bulk boundary condition.

With these boundary conditions it is possible to get a rigorous proof of the existence of the solution and its convergence as $\varepsilon \to 0$ to the solution of OBE. The proof will be presented elsewhere. The first step is the following

Proposition 3.1. *Suppose that there is $t_0 > 0$ such that $p(t)$, $\theta(t)$ and $u(t)$ are smooth solutions of OBE, with $\|\nabla u(t)\|_{H_s} + \|\nabla\theta(t)\|_{H_s} \le q$ for sufficiently large s and $t \le t_0$. Then it is possible to determine functions f_n, $n = 2,\dots,5$ satisfying, for $0 < t \le t_0$, equation (3.11) and the conditions*

$$
f_n(0,\underline{x},v) = f_n^0, \qquad f_n(t,x,y,\pm 1, v_x, v_y, v_z \ \ 0 \ \ ) = \alpha_n^{\pm} M_\pm, \quad t > 0, \tag{3.27}
$$

$$
\langle A\rangle = 0 \tag{3.28}
$$

$$\int_{^3\times^2\times[-1,1]} dvdxdydzf_n = 0 \tag{3.29}$$

Moreover, for any $\ell \geq 3$ there is a constant c such that:

$$|f_n|_{\ell,h} < cq \tag{3.30}$$

$$|A|_{\ell,h} < cq, \tag{3.31}$$

for $h \leq 1/(4T_+)$. Here

$$|f|_{\ell,h} = \sup_{\underline{x}\in^2\times[-1,1]} \sup_{v\in^3} (1+|v|)^\ell \exp[hv^2]|f(y,v)| \tag{3.32}$$

To complete the construction of a solution to BE, we have to show that the remainder is bounded in norm $|\cdot|_{\ell,h}$. The remainder has to satisfy (3.13) and the conditions

$$R(t,x,y,\pm 1,v) = \alpha_R^{\pm}\overline{M}_{\pm}, \quad \text{for } t>0; \quad R(0,\underline{x},v) = 0. \tag{3.33}$$

Moreover R has to satisfy

$$\langle v_z R\rangle = 0 \;\text{ in } z = \pm 1 \tag{3.34}$$

that implies $\alpha_R^{\pm} = \pm\int_{v_z 0} v_z R(t,\pm 1,v)dv$. To construct the solution of (3.13),(3.33), we first deal with the following linear initial boundary value problem: given D on $^2\times[-1,1]\times$ ^{3}, find R such that

$$\partial_t R + \varepsilon^{-1} v\cdot\nabla + G\cdot\nabla_v R = \varepsilon^{-2}\mathcal{L}R + \varepsilon^{-1}\mathcal{L}^1 R + \mathcal{L}^2 R + D, \tag{3.35}$$

Once one obtains good estimates for the solution of this linear problem, the non linear problem is solved by simple Banach fixed point arguments, for small q and ε. Namely, we can prove the following

Proposition 3.2. *There are ε_0, and q_0 such that, if $\varepsilon < \varepsilon_0$ and $q < q_0$, for $0 < t \leq t_0$ there is a unique solution to the initial boundary value problem* (3.13), (3.33) *verifying the following: for any positive integer ℓ there is a constant $c > 0$ such that*

$$|R|_{\ell,h} \leq ce^{ct_0}\varepsilon^{\frac{1}{2}}|A|_{\ell,h} \tag{3.36}$$

for any $h \leq 1/(4T_+)$.

This allows to conclude the existence of the solution f^ε and its convergence to the solution of the OBE in the norm (3.32).

Remark: In the case $G = 0$ the bulk boundary conditions may be replaced by the true boundary conditions and the result is still true.

We conclude by remarking that the main question to solve for the Benard problem is about the existence, stability and attractivity of stationary solutions of the boundary value problem. Unfortunately we have not so much to say about that. We can only state the following proposition about the existence of stationary solutions.

Proposition 3.3. *Let M_s be the Maxwellian with parameters ρ_s, T_s and vanishing mean velocity. Then there are $\lambda_0 > 0$ and $\varepsilon_0 > 0$ such that, if $\lambda < \lambda_0$ and $\varepsilon < \varepsilon_0$, there is a stationary solution to the bulk boundary value problem of the form* (3.8).

We note explicitly that the condition on λ corresponds to assume small Rayleigh number, so that at hydrodynamical level there is just the purely conductive solution. At the moment we have no results for the convective solutions which should appear for larger values of the Rayleigh number. The proof follows by argument similar to the ones presented in [20], [21] to which we refer for more details.

4. Microscopic description.

A system of many interacting particles, moving according to the Newton equations of motion, can be described on a space scale much larger than the typical microscopic scale (say the range of the interaction) in terms of density, velocity and temperature fields, satisfying hydrodynamic equations. In fact, on the macroscopic scale the quantities which are locally conserved (slow modes) play a major role in the motion of the fluid. The derivation of the Euler equations is based on the assumption of local equilibrium. On times of order ε^{-1}, the system is expected to be described approximately by a local Gibbs measure, with parameters (the hydrodynamical fields) varying on regions of order ε^{-1}, if ε is a scale parameter. The local equilibrium assumption implies that the parameters of the local Gibbs measure satisfy the Euler equations [23], [24]. The microscopic locally conserved quantities converge, as $\varepsilon \to 0$, by a law of large numbers, to the macroscopic fields. To make this correct, the many particles Hamiltonian system must have good dynamical mixing properties to approach and stay in a state close to the local equilibrium. At the moment it is not understood how to provide such properties. Therefore the only rigorous results are obtained by adding some noise to the Hamiltonian evolution [2] (see [25] for a review on the rigorous results for stochastic systems).

As we have explained in the previous Sections, to get the viscous terms, times of order ε^{-2} have to be considered. In [9] we gave a formal derivation of the INS from a Hamiltonian particles system under the parabolic rescaling, in the low Mach number regime.

One of the main ingredients is the assumption that the non-equilibrium density on the phase space F^ε can be expressed as a truncated series in the parameter ε, inspired by the Hilbert-Chapmann-Enskog expansion for the solution of the rescaled Boltzmann equation (see previous section for more extensive discussion of this subject). Using this expansion we can get the equations for the non-equilibrium expectation of the locally conserved observables, which are at least formally meaningful in the limit $\varepsilon \to 0$.

We consider a system of N identical particles of mass 1 in $\Omega^{(\varepsilon)} = \varepsilon^{-1 2} \times [-\varepsilon^{-1}, \varepsilon^{-1}]$, interacting through a pair central potential V of finite range. Moreover the system is subject to an external force of order ε^2 coming from a potential h. After rescaling space as ε^{-1} and time as ε^{-2} the Newton equations become

$$\frac{dx_i}{dt}(t) = \varepsilon^{-1} v_i(t) \tag{4.1}$$

$$\frac{dv_i}{dt}(t) = -\varepsilon^{-2} \sum_{i \neq j} \nabla V(\varepsilon^{-1}(x_i - x_j)) - \sum_i \nabla h(x_i). \tag{4.2}$$

In (4.2) we used the convention that ∇ denotes always the differentiation with respect to the argument of the function to which it is applied. So the first one means differentiation of V with respect to the microscopic variable ξ which is put equal to $\varepsilon^{-1}(x_i - x_j)$ afterwards, while the second one denotes differentiation with respect to x_i.

To complete the description of the microscopic motion we have to specify what happens when a particle hits the boundary. We assume periodic boundary conditions on the boundaries $x = \pm\varepsilon^{-1}, y = \pm\varepsilon^{-1}$. On the other side, when a particle hits the boundaries $z = \pm 1$ in a point $(\bar{x}, \bar{y}, \pm 1)$ with velocity v, v_z 0, it is diffusely reflected, namely it is reflected in the same point with velocity v' (such that v'_z 0) chosen at random, with distribution density $\Phi(v')$ proportional to $|v'_z| \exp -(v')^2/T_\pm$. In this way the trajectory of the system of n particles in the phase space is a stochastic process sampled by the initial conditions and the random choice of the outgoing velocities of the particles hitting the boundary. For fixed N this stochastic process can be shown to be well defined for almost all initial data with respect to the Liouville measure and almost all the outgoing velocities with respect to $\Phi(v')$. This follows along the same lines of [26] where elastic reflection is considered. The number of particles N is assumed to be of order ε^{-3} to keep the density finite. This has to be compared with the kinetic description given by the Grad-Boltzmann limit ($N \sim \varepsilon^{-2}$), which becomes correct in the limit of vanishing density. The total number of particles, the three components of the total momentum and the total energy are the conserved quantities. We construct the corresponding empirical fields:

empirical density

$$z^0(x) = \varepsilon^3 \sum_i \delta(x_i - x) \tag{4.3}$$

empirical velocity field density

$$z^\alpha(x) = \varepsilon^3 \sum_i v_i^\alpha \delta(x_i - x), \quad \alpha = 1, \dots, 3 \tag{4.4}$$

empirical energy density

$$z^4(x) = \varepsilon^3 \sum_i \frac{1}{2}\big[v_i^2 + \varepsilon 2h(x_i) + \sum_{j \neq i} V(\varepsilon^{-1}|x_i - x_j|)\big]\delta(x_i - x) \tag{4.5}$$

We will write also

$$z^\mu(x) = \varepsilon^3 \sum_i \delta(x_i - x) z_i^\mu \tag{4.6}$$

with $z_i^0 = 1$; $z_i^\alpha = v_i^\alpha, \alpha = 1, \dots, 3$; $z_i^4 = \frac{1}{2}[v_i^2 + \sum_{i \neq j} V(\varepsilon^{-1}|x_i - x_j|) + 2\varepsilon h(x_i)]$. The generalized functions z^α on the phase space are expected to be approximated, to the lowest order in ε, by the macroscopic hydrodynamic fields, in the sense that, with probability 1, for any smooth function f, we have

$$\int dx z(x) f(x) = \int dx b(x) f(x) + o(1) \tag{4.7}$$

where $o(1)$ denotes a quantity going to 0 as $\varepsilon \to 0$, $z = \{z^\alpha\}$ and $b = \{\rho, U, e\}$, the macroscopic density, velocity field and energy respectively. The empirical fields satisfy the following local conservation laws, which are obtained differentiating $z^\alpha(x,t)$ with respect to the time and using the Newton equations:

$$\frac{\partial}{\partial t}\int dx f(x) z^\beta(x) = \varepsilon^{-1}\int dx \sum_{k=1}^{3} \frac{\partial f}{\partial x^k} w^{\beta k}(x) + O(\varepsilon) \tag{4.8}$$

where $w^{\beta k}$, $\beta = 0,\ldots,4$; $k = 1,\ldots,3$ are the currents associated to the fields z^β and are explicitly given by

$$w^{0k}(x) = \varepsilon^3 \sum_i \delta(x_i - x) v_i^k \tag{4.9}$$

$$w^{\beta k}(x) = \varepsilon^3 \sum_i \delta(x_i - x)\{v_i^\beta v_i^k + \frac{1}{2}\sum_j \Psi^{\beta k}(\varepsilon^{-1}(x_i - x_j)) + \varepsilon h(x_i)\}, \quad \beta = 1,\ldots,3 \tag{4.10}$$

$$w^{4k}(x) = \varepsilon^3 \sum_i \{v_i^k z_i^4 + [\frac{1}{2}\sum_{j,\gamma} \Psi^{\gamma k}(\varepsilon^{-1}(x_i - x_j)) + \varepsilon h(x_i)]\frac{1}{2}[v_i^\gamma + v_j^\gamma]\} \tag{4.11}$$

where $\Psi^{\gamma k}(\xi) = -\nabla_\beta V(\xi)\xi^\gamma$. We put also $w^{\beta k}(x) = \varepsilon^3 \sum_i \delta(x_i - x) w_i^{\beta k}$.
$z^\alpha(x)$ are approximate integrals of the motion in the sense that $\mathcal{L}\left[\varepsilon^3 \sum_i f(\varepsilon\xi_i) z_i^\alpha\right] = O(\varepsilon)$, where $\mathcal{L}$ is the Liouville operator and $\xi_i = \varepsilon^{-1} x_i$ are the microscopic coordinates. The local equilibrium distribution on the phase space (in microscopic variables) is $G = Z^{-1}\exp\sum_i \sum_{\alpha=0}^{4} \lambda^\alpha(\varepsilon\xi_i) z_i^\alpha$, with Z the normalization factor, and satisfies $\mathcal{L}G = O(\varepsilon)$. We need ergodic properties of the Liouville operator in order that the system approaches and stays in a state close to the local equilibrium. We assume the following property for the Liouville operator:

local ergodicity: the space of the invariant observables for the microscopic dynamics reduces to the locally conserved quantities, mass, momentum and energy.

We look for a solution to the rescaled Liouville equation

$$\frac{\partial F_\varepsilon}{\partial t} = \varepsilon^{-2}\mathcal{L}^\star F_\varepsilon \tag{4.12}$$

where $\mathcal{L}^\star$ is the adjoint, w.r.t. the Liouville measure, of $\mathcal{L}$, formally given by $\mathcal{L}^\star = -\mathcal{L}$.

Writing F_ε as a series in ε, $F_\varepsilon = \sum_n \varepsilon^n F^n$, and substituting it in (4.12) we would get the diverging terms $\varepsilon^{-2}\mathcal{L}^\star F_0$ and $\varepsilon^{-1}\mathcal{L}^\star F_1$. Therefore we are forced to put $\mathcal{L}^\star F_0 = 0$, hence

F_0 has to be the global equilibrium. Moreover $\varepsilon^{-1}\mathcal{L}^\star F_1$ is not divergent if $\mathcal{L}^\star F_1 = O(\varepsilon)$. By the local erdocity assumption, this means that the term of order ε has to be a function only of the empirical fields.

To single out the non-hydrodynamic contribution to F_ε let us write F_ε as a part which is Gibbsian with parameters slowly depending on the microscopic variables and depending on ε by means of a series in ε, and a remainder. More explicitly, we put

$$F_\varepsilon = G_\varepsilon + \varepsilon^2 G_0 R_\varepsilon \tag{4.13}$$

with

$$G_\varepsilon = Z_\varepsilon^{-1}\exp\{\sum_{i,\mu}\lambda_\varepsilon^\mu(x_i,t)z_i^\mu\}; \quad \lambda_\varepsilon^\mu(x,t) = \sum_{n=0}^{\infty}\varepsilon^n\lambda_n^\mu(x_i,t); \quad \lambda_0^\mu = \text{const.} \tag{4.14}$$

G_0 is the zero order term in the expansion, the global equilibrium. In this way we have included all the hydrodynamic terms in G_ε and we can assume that in R_ε there are no terms which are combinations of the invariant quantities z^α with coefficients depending on the macroscopic variables, since these terms are already present in G_ε.

We put

$$R_\varepsilon(t) = R(t) + O(\varepsilon) \tag{4.15}$$

In [9] it is shown that R can be determined as solution to the equation:

$$\int_0^t [\mathcal{L}^\star R - \sum_i\sum_{\mu,\gamma}\frac{\partial\lambda_1^\mu}{\partial x_i^\gamma}(x_i,s)w_i^{\mu\gamma}] = 0 \tag{4.16}$$

If we assume that there exists a solution $R(t)$ to (4.16), then R is expressed in terms of the empirical fields as $\mathcal{L}^{\star -1}\sum_i\sum_{\mu,\gamma}\frac{\partial\lambda_1^\mu}{\partial x_i^\gamma}(x_i,t)w_i^{\mu\gamma}$. In this way, by inserting (4.13) in the conservation laws averaged with respect to F_ε, we can get closed equations for the empirical fields up to order ε. In [9], for the case $h = 0$, it has been possible to derive the incompressible Navier-Stokes equations under above assumptions.

We specialize now to the case of the gravity, namely $h(x) = -G\cdot\underline{x}$ with $G = (0,0,-g)$. We make the same assumptions as before on the Liouville operator and assume expression (4.13) for F^ε . By the procedure in [9], we get the OBE, the only difference being that we have to compute also the terms involving the force. As an example, we show how to get the Boussinesq condition. By the conservation laws (4.8) for $\beta = 1 \ldots 3$, averaged versus F^ε, integrating on time we get

$$\int dx f(x)[\rho_\varepsilon u_\varepsilon(x,t) - \rho_\varepsilon u_\varepsilon(x,0)] = \int_0^t ds \int dx \frac{\partial f}{\partial_{x^k}}(x)\langle \varepsilon^{-1} w^{\beta k}\rangle_{F_\varepsilon}$$

where $\rho_\varepsilon u_\varepsilon^\alpha =< z^\alpha >_{F_\varepsilon}$. In the limit $\varepsilon \to 0$ the l.h.s. vanishes. On the other hand

$$\varepsilon^{-1}\langle w^{\beta k}\rangle_{F_\varepsilon} = \varepsilon^{-1}\langle w^{\beta k}\rangle_{G_\varepsilon} + \varepsilon\langle w^{\beta k} R_\varepsilon\rangle$$

We decompose $w^{\beta k}$ as

$$w_i^{\beta k} = \tilde{w}_i^{\beta k} + \varepsilon^2 u^k(x_i)u^\beta(x_i) + \varepsilon u^k(x_i)\tilde{v}_i^\beta + \varepsilon u^\beta(x_i)\tilde{v}_i^k + \varepsilon h(x_i)$$

For the symmetry of the measure G_ε we have $\left\langle \tilde{w}^{\beta k}(x)\right\rangle_{G_\varepsilon} = O(\varepsilon^4)$, if $k \neq \beta$. The average of $\tilde{w}^{\beta\beta}$, $\beta = 1, \ldots, d$, with respect the local Gibbs state G_ε is, by the virial theorem [25], the thermodynamic pressure P_ε in the state G_ε. Since $h = -G \cdot \underline{x}$ we get

$$\varepsilon^{-1} \int dx f(x) \nabla[P^\varepsilon(x,s) - \varepsilon\rho_+ G \cdot \underline{x}] = O(\varepsilon) \tag{4.17}$$

Since P^ε is a function of the thermodynamic parameters λ_ε, we can expand it in series of ε as $\sum_k \varepsilon^k P_k$, where $P_k = \frac{d^k P^\varepsilon}{d\varepsilon^k}|_{\varepsilon=0}$ We have that P_0 is constant since it is a function of the constants λ_0^0 and λ_0^4, while $P_1 = \sum_{\mu=0}^4 \frac{\partial P^\varepsilon}{\partial \lambda_\varepsilon^\mu}|_{\varepsilon=0}\lambda_1^\mu$. In order to fulfil (4.17) for any test function f, $P_1 - \rho_+ G \cdot \underline{x}$ has to be constant in the limit. Recalling the notations of Sect. 2, where P_s and T_s are the pressure and temperature solution of the stationary equations (2.6) and $\varepsilon\tilde{P}_1 = P_\varepsilon - P_s + O(\varepsilon^2)$, we get

$$\nabla \tilde{P}_1 = 0$$

and the relation with the fluctuations of density and temperature r and θ with respect to the stationary profile is given by

$$\tilde{P}_1 = c_r r + c_\theta \theta = 0$$

where $c_r = \frac{\partial P_\varepsilon}{\partial \rho}\big|_0$ and $c_\theta = \frac{\partial P_\varepsilon}{\partial T}\big|_0$. The label 0 here means that one has to put $\varepsilon = 0$ afterwards. As a consequence, the expression for the thermal expansion coefficient α in (2.11) will be $\alpha = (c_r\,\rho)^{-1} c_\theta$.

The other equations of the Boussinesq system are obtained as in [9] and we refer to that paper for details. The transport coefficients, bulk and shear viscosity and the conductivity are given by the usual Green-Kubo formulas. The correlations of the currents appearing in these formulas are computed at the global equilibrium, hence the transport coefficients are independent of x and t. Therefore also the assumptions that these coefficients are independent of the temperature in the Boussinesq approximation is derived in this approach and is a consequence only of the scaling.

The rigorous proof of above statements is out of the possibilities of the present knowledge. Rigorous results in the direction here discussed are available for a much simpler model, namely the simple exclusion process [10]. This is a stochastic system of particle on the lattice interacting via an hard core force that prevents more than one particle per site. Particles jump to unoccupied sites with intensity depending on the jump. The only conserved quantity is the density. The analog of the incompressible limit in this case is obtained considering initially deviations from the constant density profile of order ε. In [10] it is proved that in space dimension bigger that 2 and for product initial states, the deviation from the constant density stays of order ε at later times and it converges weakly in probability to the solution of the viscous Burgers equation with a diffusion matrix given by the Green-Kubo formula for this model.

Acknowledgments.

It is a pleasure to thank the Department of Information Science of the Tokyo Institute of Technology for the warm hospitality. One of us (R.E.) also thanks the JSPS for financial support.

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